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WHEELS, WHEELING, AND THE KONTSEVICH INTEGRAL OF THE UNKNOT*

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ABSTRACT

We conjecture an exact formula for the Kontsevich integral of the unknot, and also conjecture a formula (also conjectured independently by Deligne [De]) for the relation between the two natural products on the space of uni-trivalent diagrams. The two formulas use the related notions of "Wheels" and "Wheeling". We prove these formulas 'on the level of Lie algebras' using standard techniques from the theory of Vassiliev invariants and the theory of Lie algebras. In a brief epilogue we report on recent proofs of our full conjectures, by Kontsevich [Ko2] and by DBN, DPT, and T. Q. T. Le, [BLT].

CONTENTS

1. Introduction

1.1. THE CONJECTURES. Let us start with the statements of our conjectures; the rest of the paper is concerned with motivating and justifying them. We assume some familiarity with the theory of Vassiliev invariants. See e.g. [B-N1, Bi, BL, Go1, Go2, Ko1, Vas1, Vas2] and [B-N2].

Very briefly, recall that any complex-valued knot invariant V can be extended to an invariant of knots with double points (**singular knots**) via the formula $V(\times) = V(\nearrow\!\!\!\!\nwarrow) - V(\nwarrow\!\!\!\!\nearrow)$. An invariant of knots (or framed knots) is called a **Vassiliev invariant**, or a **finite type invariant of type** m, if its extension to singular knots vanishes whenever evaluated on a singular knot that has more than

m double points. Vassiliev invariants are in some senses analogues to polynomials (on the space of all knots), and one may hope that they separate knots. While this is an open problem and the precise power of the Vassiliev theory is yet unknown, it is known (see [Vo]) that Vassiliev invariants are strictly stronger than the Reshetikhin–Turaev invariants ([RT]), and in particular they are strictly stronger than the Alexander–Conway, Jones, HOMFLY, and Kauffman invariants. Hence one is interested in a detailed understanding of the theory of Vassiliev invariants.

The set $\mathcal{V}$ of all Vassiliev invariants of framed knots is a linear space, filtered by the type of an invariant. The fundamental theorem of Vassiliev invariants, due to Kontsevich [Ko1], says that the associated graded space $\operatorname{gr}\mathcal{V}$ of $\mathcal{V}$ can be identified with the graded dual $\mathcal{A}^\star$ of a certain completed graded space $\mathcal{A}$ of formal linear combinations of certain diagrams, modulo certain linear relations. The diagrams in $\mathcal{A}$ are connected graphs made of a single distinguished directed line (the **skeleton**), some number of undirected **internal edges**, some number of trivalent **external vertices** in which an internal edge ends on the skeleton, and some number of trivalent **internal vertices** in which three internal edges meet. It is further assumed that the internal vertices are **oriented**: that for each internal vertices one of the two possible cyclic orderings of the edges emanating from it is specified. An example of a diagram in $\mathcal{A}$ is in Figure 1. The linear relations in the definition of $\mathcal{A}$ are the well-known AS, IHX, and STU relations, also shown in Figure 1. The space $\mathcal{A}$ is graded by half the total number of trivalent vertices in a given diagram.

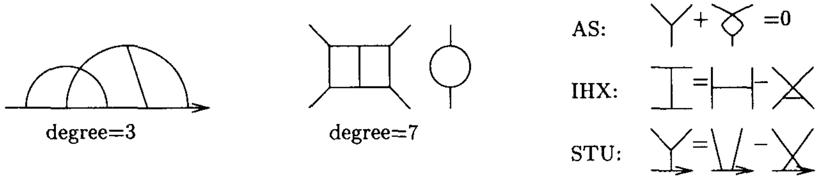


Figure 1. A diagram in $\mathcal{A}$, a diagram in $\mathcal{B}$ (a uni-trivalent diagram), and the AS, IHX, and STU relations. All internal vertices shown are oriented counterclockwise.

The most difficult part of the currently known proofs of the isomorphism $\mathcal{A}^\star \cong \operatorname{gr}\mathcal{V}$ is the construction of a **universal Vassiliev invariant**: an $\mathcal{A}$-valued framed-knot invariant ζ that satisfies a certain universality property which implies that its adjoint $\zeta^\star\colon \mathcal{A}^\star \to \mathcal{V}$ is well defined and induces an isomorphism $\mathcal{A}^\star \cong \operatorname{gr}\mathcal{V}$, as required (see e.g. [BS]). Such a universal Vassiliev invariant is not unique; the set of universal Vassiliev invariants is in a bijective correspondence with the set of all filtration-respecting maps $\mathcal{V} \to \operatorname{gr}\mathcal{V}$ that induce the identity map $\operatorname{gr}\mathcal{V} \to \operatorname{gr}\mathcal{V}$. But it is a noteworthy and not terribly well understood fact that all known constructions of a universal Vassiliev invariant are either known to give the same answer or are conjectured to give the same answer as the **framed Kontsevich integral** Z (see Section 2.2), the first universal Vassiliev invariant ever constructed. Furthermore, the Kontsevich integral is well behaved in several senses, as shown in [B-N1, BG, Kas, Kol, LMMO, LM1, LM2].

Thus it seems that Z is a canonical and not an accidental object. It is therefore surprising how little we know about it. While there are several formulas for computing Z, they are all of limited use beyond the first few degrees. Before this paper was written, no explicit formula for the value of Z on *any* knot was known, not even the unknot!

Our first conjecture is about the value of the Kontsevich integral of the unknot. We conjecture a completely explicit formula, written in terms of an alternative realization of the space $\mathcal{A}$, the space $\mathcal{B}$ of **uni-trivalent diagrams** ("Chinese characters", in the language of [B-N1]). The space $\mathcal{B}$ is also a completed graded space of formal linear combinations of diagrams modulo linear relations: the diagrams are the so-called uni-trivalent diagrams, which are the same as the diagrams in $\mathcal{A}$ except that a skeleton is not present, and instead a certain number of univalent vertices are allowed (the original connectivity requirement is dropped, but one insists that every connected component of a uni-trivalent diagram would have at least one univalent vertex). An example of a uni-trivalent diagram is in Figure 1. The relations are the AS and IHX relations that appear in the same figure (but not the STU relation, which involves the skeleton). The degree of a uni-trivalent diagram is half the total number of its vertices. There is a natural isomorphism $\chi\colon \mathcal{B} \to \mathcal{A}$ which maps every uni-trivalent diagram to the average of all possible ways of placing its univalent vertices along a skeleton line (see [B-N1], but notice that a sum is used there instead of an average). In a sense that we will recall below, the fact that χ is an isomorphism is an analog of the Poincare–Birkhoff–Witt (PBW) theorem. We note that the inverse map σ of χ is more difficult to construct and manipulate.

CONJECTURE 1 (Wheels): *The framed Kontsevich integral of the unknot, $Z(\bigcirc)$, expressed in terms of uni-trivalent diagrams, is equal to*

$$\Omega = \exp_{\cup} \sum_{n=1}^{\infty} b_{2n}\omega_{2n}. \tag{1}$$

The notation in (1) means:

- The 'modified Bernoulli numbers' b_{2n} are defined by the power series expansion

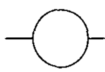
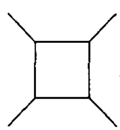
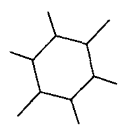
$$\sum_{n=0}^{\infty} b_{2n}x^{2n} = \frac{1}{2}\log\frac{\sinh x/2}{x/2}. \tag{2}$$

These numbers are related to the usual Bernoulli numbers B_{2n} and to the values of the Riemann ζ-function on the even integers via (see e.g. [Ap, Section 12.12])

$$b_{2n} = \frac{B_{2n}}{4n(2n)!} = \frac{(-1)^{n+1}}{2n(2\pi)^{2n}}\zeta(2n).$$

The first three modified Bernoulli numbers are $b_2 = 1/48$, $b_4 = -1/5760$, and $b_6 = 1/362880$.

- The '$2n$-wheel' ω_{2n} is the degree $2n$ uni-trivalent diagram made of a $2n$-gon with $2n$ legs:

$\omega_2 =$ , $\omega_4 =$ , $\omega_6 =$ , $\dots,$

(with all vertices oriented counterclockwise).* We note that the AS relation implies that odd-legged wheels vanish in $\mathcal{B}$, and hence we do not consider them.

- $\exp_{\sqcup}$ means 'exponential in the disjoint union sense'; that is, it is the formal-sum exponential of a linear combination of uni-trivalent diagrams, with the product being the disjoint union product.

Let us explain why we believe the Wheels Conjecture (Conjecture 1). Recall ([B-N1]) that there is a parallelism between the space $\mathcal{A}$ (and various variations thereof) and a certain part of the theory of Lie algebras. Specifically, given a metrized Lie algebra $\mathfrak{g}$, there exists a commutative square (a refined version is in

* Wheels have appeared in several noteworthy places before: [Ch, CV, KSA, Vai]. Similar but slightly different objects appear in Ng's beautiful work on ribbon knots [Ng].

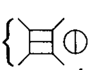
Theorem 3 below):

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\mathcal{T}_\mathfrak{g}"] & U^\mathfrak{g}(\mathfrak{g}) & \left(\text{the } \mathfrak{g}\text{-invariant part of the completed universal enveloping algebra of } \mathfrak{g}\right) \\ \mathcal{B} \arrow[u, "\chi", "\wr"'] \arrow[r, "\mathcal{T}_\mathfrak{g}"] & S^\mathfrak{g}(\mathfrak{g}) \arrow[u, "\beta_\mathfrak{g}", "\wr"'] & \left(\text{the } \mathfrak{g}\text{-invariant part of the completed symmetric algebra of } \mathfrak{g}\right) \end{tikzcd} \tag{3}$$

In this square the left column is the above mentioned formal PBW isomorphism χ, and the right column is the symmetrization map $\beta_\mathfrak{g}$: $S(\mathfrak{g}) \to U(\mathfrak{g})$, sending an unordered word of length n to the average of the $n!$ ways of ordering its letters and reading them as a product in $U(\mathfrak{g})$. The map $\beta_\mathfrak{g}$ is a vector space isomorphism by the honest PBW theorem. The left-to-right maps $\mathcal{T}_\mathfrak{g}$ are defined as in [B-N1] by contracting copies of the structure constants tensor, one for each vertex of any given diagram, using the standard invariant form $(\cdot,\cdot)$ on $\mathfrak{g}$ (see citations in section 2.2 below). The maps $\mathcal{T}_\mathfrak{g}$ seem to 'forget' some information (some high-degree elements on the left get mapped to 0 on the right no matter what the algebra $\mathfrak{g}$ is, see [Vo]), but at least up to degree 12 they are faithful (for some Lie algebras); see [Kn].

THEOREM 1: *Conjecture 1 is "true on the level of semi-simple Lie algebras". Namely,*

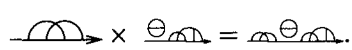
$$\mathcal{T}_\mathfrak{g}\Omega = \mathcal{T}_\mathfrak{g}\chi^{-1}Z(\bigcirc).$$

We now formulate our second conjecture. Let $\mathcal{B}' = \operatorname{span}\left\{ \right.$  $\left.\right\} /(AS, IHX)$ be the same as $\mathcal{B}$, only dropping the remaining connectivity requirement so that we also allow connected components that have no univalent vertices (but each with at least one trivalent vertex). The space $\mathcal{B}'$ has two different products, and thus is an algebra in two different ways:

- The disjoint union $C_1 \dot\cup C_2$ of two uni-trivalent diagrams $C_{1,2}$ is again a uni-trivalent diagram. The obvious bilinear extension of $\dot\cup$ is a well defined product $\mathcal{B}' \times \mathcal{B}' \to \mathcal{B}'$, which turns $\mathcal{B}'$ into an algebra. For emphasis we will call this algebra $\mathcal{B}'_{\dot\cup}$.
- $\mathcal{B}'$ is isomorphic (as a vector space) to the space

$$\mathcal{A}' = \operatorname{span}\{ \ \}/(AS, IHX, STU)$$

of diagrams whose skeleton is a single oriented interval (like $\mathcal{A}$, only that here we also allow non-connected diagrams). The isomorphism is the map χ: $\mathcal{B}' \to \mathcal{A}'$ that maps a uni-trivalent diagram with k "legs" (univalent vertices) to the average of the $k!$ ways of arranging them along an oriented interval (in [B-N1] the sum was used instead of the average). $\mathcal{A}'$ has a well known "juxtaposition" product $\times$, related to the "connect sum" operation on knots:

(4) 

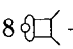
The algebra structure on $\mathcal{A}'$ defines another algebra structure on $\mathcal{B}'$. For emphasis we will call this algebra $\mathcal{B}'_\times$.

As before, $\mathcal{A}'$ is graded by half the number of trivalent vertices in a diagram, $\mathcal{B}'$ is graded by half the total number of vertices in a diagram, and the isomorphism χ as well as the two products respect these gradings.

Definition 1.1: If C is a uni-trivalent diagram without struts (components like ), let $\hat{C}$: $\mathcal{B}' \to \mathcal{B}'$ be the operator defined by

$$\hat{C}(C') = \begin{cases} 0 & \text{if } C \text{ has more legs than } C', \\ \text{the sum of all ways of gluing} & \\ \text{all the legs of } C \text{ to some (or} & \\ \text{all) legs of } C' & \text{otherwise.} \end{cases}$$

For example,

$$\widehat{\omega_4}(\omega_2) = 0; \quad \widehat{\omega_2}(\omega_4) = 8$$  $+ 4$ .

If C has k legs and total degree m, then $\hat{C}$ is an operator of degree $m - k$. By linear extension, we find that every $C \in \mathcal{B}'$ without struts defines an operator $\hat{C}$: $\mathcal{B}' \to \mathcal{B}'$. (We restrict to diagrams without struts to avoid circles arising from the pairing of two struts and to guarantee convergence.)

As Ω is made of wheels, we call the action of the (degree 0) operator $\hat{\Omega}$ "wheeling". As Ω begins with 1, the wheeling map is invertible. We argue below that $\hat{\Omega}$ is a diagrammatic analog of the Duflo isomorphism $S^{\mathfrak{g}}(\mathfrak{g}) \to S^{\mathfrak{g}}(\mathfrak{g})$ (see [Du] and see below). The Duflo isomorphism intertwines the two algebra structures that $S^{\mathfrak{g}}(\mathfrak{g})$ has: the structure it inherits from the symmetric algebra and the structure it inherits from $U^{\mathfrak{g}}(\mathfrak{g})$ via the PBW isomorphism. One may hope that $\hat{\Omega}$ has the parallel property:

CONJECTURE 2 (Wheeling*): *Wheeling intertwines the two products on uni-trivalent diagrams. More precisely, the map* $\hat{\Omega}$: $\mathcal{B}'_{\cup} \to \mathcal{B}'_{\times}$ *is an algebra isomorphism.*

There are several good reasons to hope that Conjecture 2 is true. If it is true, one would be able to use it along with Conjecture 1 and known properties of the Kontsevich integral (such as its behavior under the operations of change of framing, connected sum, and taking the parallel of a component as in [LM2]) to get explicit formulas for the Kontsevich integral of several other knots and links. Note that change of framing and connect sum act on the Kontsevich integral multiplicatively using the product in $\mathcal{A}$, but the conjectured formula we have for the Kontsevich integral of the unknot is in $\mathcal{B}$. Using Conjecture 2 it should be possible to perform all operations in $\mathcal{B}$. Likewise, using Conjectures 1 and 2 and the hitherto known or conjectured values of the Kontsevich integral, one would be able to compute some values of the LMO 3-manifold invariant [LMO], using the "Århus integral" formula of [Å-I, Å-II, Å-III].

Perhaps a more important reason is that, in essence, $\mathcal{A}$ and $\mathcal{B}$ capture that part of the information about $U(\mathfrak{g})$ and $S(\mathfrak{g})$ that can be described entirely in terms of the bracket and the structure constants. Thus a proof of Conjecture 2 would yield an elementary proof of the intertwining property of the Duflo isomorphism, whose current proofs use representation theory and are quite involved. We feel that the knowledge missing to give an elementary proof of the intertwining property of the Duflo isomorphism is the same knowledge that is missing to give a proof of the Kashiwara–Vergne conjecture ([KV]).

THEOREM 2: *Conjecture 2 is "true on the level of semi-simple Lie algebras". A precise statement is in Proposition 2.1 and the remark following it.*

Remark 1.2: As semi-simple Lie algebras "see" all of the Vassiliev theory at least up to degree 12 [B-N1, Kn], Theorems 1 and 2 imply Conjectures 1 and 2 up to that degree. It should be noted that semi-simple Lie algebras do not "see" the whole Vassiliev theory at high degrees, see [Vo].

Remark 1.3: We've chosen to work over the complex numbers to allow for some analytical arguments below. The rationality of the Kontsevich integral [LM1] and the uniform classification of semi-simple Lie algebras over fields of characteristic 0 implies that Conjectures 1 and 2 and Theorems 1 and 2 are independent of the (characteristic 0) ground field.

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* Conjectured independently by Deligne [De].

1.2. THE PLAN. Theorem 1 and Theorem 2 both follow from a delicate assembly of widely known facts about Lie algebras and related objects; the main novelty in this paper is the realization that these known facts can be brought together and used to prove Theorems 1 and 2 and make Conjectures 1 and 2. The facts we use about Lie-algebras amount to the commutativity of a certain monstrous diagram. In Section 2 below we will explain everything that appears in that diagram, prove its commutativity, and prove Theorem 2. In Section 3 we will show how that commutativity implies Theorem 1 as well. We conclude this introductory section with a picture of the monster itself:

THEOREM 3 (definitions and proof in Section 2): *The following monster diagram is commutative:*

$$\begin{tikzcd}
\mathcal{K}^F \arrow[d, "Z"'] \arrow[drr, "RT_{\mathfrak{g}}"] \arrow[dr, phantom, "\boxed{1}"] & & & & & \\
\mathcal{A}' \arrow[rr, "\mathcal{T}^{\hbar}_{\mathfrak{g}}"] \arrow[drr, phantom, "\boxed{2}"] & & U(\mathfrak{g})^{\mathfrak{g}}[[\hbar]] \arrow[rr, "\sim", "\psi_{\mathfrak{g}}"'] \arrow[drr, phantom, "\boxed{3}"] & & P(\mathfrak{h}^\star)^W[[\hbar]] & \\
\mathcal{B}'_{\times} \arrow[u, "\chi", "\wr"'] \arrow[rr, "\mathcal{T}^{\hbar}_{\mathfrak{g}}"] \arrow[drr, phantom, "\boxed{4}"] & & S(\mathfrak{g})^{\mathfrak{g}}_{\times}[[\hbar]] \arrow[u, "\beta_{\mathfrak{g}}", "\wr"'] \arrow[rr, equal] \arrow[drr, phantom, "\boxed{5}"] & & P(\mathfrak{g}^\star)^{\mathfrak{g}}[[\hbar]] & \\
& \mathcal{B}'_{\cup} \arrow[ul, "\hat{\Omega}", "\wr"'] \arrow[rr, "\mathcal{T}^{\hbar}_{\mathfrak{g}}"] & & S(\mathfrak{g})^{\mathfrak{g}}_{\cup}[[\hbar]] \arrow[ul, "D(j^{1/2}_{\mathfrak{g}})", "\wr"'] \arrow[rr, equal] & & P(\mathfrak{g}^\star)^{\mathfrak{g}}[[\hbar]] \arrow[ul, "S_{\mathfrak{g}}", "\wr"'] \arrow[uul, "\iota_{\mathfrak{g}}"', "\wr"]
\end{tikzcd}$$

Remark 1.4: Our two conjectures ought to be related—one talks about Ω, and another is about an operator $\hat{\Omega}$ made out of Ω, and the proofs of Theorems 1 and 2 both use the Duflo map ($D(j^{1/2}_{\mathfrak{g}})$ in the above diagram). But looking more closely at the proofs below, the relationship seems to disappear. The proof of Theorem 2 uses only the commutativity of the face labeled $\boxed{4}$, while the proof of Theorem 1 uses the commutativity of all faces but $\boxed{4}$. No further relations between the conjectures are seen in the proofs of our theorems. Why is it that the same strange combination of uni-trivalent diagrams Ω plays a role in these two seemingly unrelated affairs? See the epilogue (Section 4) for a partial answer.

1.3. ACKNOWLEDGEMENT. Much of this work was done when the four of us were visiting Århus, Denmark, for a special semester on geometry and physics, in August 1995. We wish to thank the organizers, J. Dupont, H. Pedersen, A. Swann and especially J. Andersen for their hospitality and for the stimulating atmosphere they created. We wish to thank the Institute for Advanced Studies for their hospitality, and P. Deligne for listening to our thoughts and sharing his. His letter [De] introduced us to the Duflo isomorphism; initially our proofs relied more heavily on the Kirillov character formula. A. Others and A. Referee made some very valuable suggestions; we thank them and also thank J. Birman, V. A. Ginzburg, A. Haviv, A. Joseph, G. Perets, J. D. Rogawski, J. D. Stasheff, M. Vergne and S. Willerton for additional remarks and suggestions.

2. The monster diagram

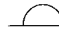
2.1. THE VERTICES. Let $\mathfrak{g}_{\mathbf{R}}$ be the (semi-simple) Lie-algebra of some compact Lie group G, let $\mathfrak{g} = \mathfrak{g}_{\mathbf{R}} \otimes \mathbf{C}$, let $\mathfrak{h} \subset i\mathfrak{g}_{\mathbf{R}}$ be a Cartan subalgebra of $\mathfrak{g}$, and let W be the Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$. Let $\Delta_+ \subset \mathfrak{h}^\star$ be a set of positive roots of $\mathfrak{g}$, and let $\rho \in i\mathfrak{g}_{\mathbf{R}}^\star$ be half the sum of the positive roots. Let $\hbar$ be an indeterminate, and let $\mathbf{C}[[\hbar]]$ be the ring of formal power series in $\hbar$ with coefficients in $\mathbf{C}$.

- $\mathcal{K}^F$ is the set of all framed knots in $\mathbf{R}^3$.
- $\mathcal{A}'$ is the algebra of not-necessarily-connected chord diagrams, as on page 218.
- $\mathcal{B}'_\times$ and $\mathcal{B}'_\sqcup$ denote the space of uni-trivalent diagrams (allowing connected components that have no univalent vertices), as on page 223, taken with its two algebra structures.
- $U(\mathfrak{g})^\mathfrak{g}[[\hbar]]$ is the $\mathfrak{g}$-invariant part of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, with the coefficient ring extended to be $\mathbf{C}[[\hbar]]$.
- $S(\mathfrak{g})^\mathfrak{g}_\times[[\hbar]]$ and $S(\mathfrak{g})^\mathfrak{g}_\sqcup[[\hbar]]$ denote the $\mathfrak{g}$-invariant part of the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$, with the coefficient ring extended to be $\mathbf{C}[[\hbar]]$. In $S(\mathfrak{g})^\mathfrak{g}_\sqcup[[\hbar]]$ we take the algebra structure induced from the natural algebra structure of the symmetric algebra. In $S(\mathfrak{g})^\mathfrak{g}_\times[[\hbar]]$ we take the algebra structure induced from the algebra structure of $U(\mathfrak{g})^\mathfrak{g}[[\hbar]]$ by the symmetrization map $\beta_\mathfrak{g}\colon S(\mathfrak{g})^\mathfrak{g}_\times[[\hbar]] \to U(\mathfrak{g})^\mathfrak{g}[[\hbar]]$, which is a linear isomorphism by the Poincare–Birkhoff–Witt theorem.
- $P(\mathfrak{h}^\star)^W[[\hbar]]$ is the space of Weyl-invariant polynomial functions on $\mathfrak{h}^\star$, with coefficients in $\mathbf{C}[[\hbar]]$.
- $P(\mathfrak{g}^\star)^\mathfrak{g}[[\hbar]]$ is the space of ad-invariant polynomial functions on $\mathfrak{g}^\star$, with coefficients in $\mathbf{C}[[\hbar]]$.

2.2. The edges.

- Z is the framed version of the Kontsevich integral for knots as defined in [LM1]. A simpler (and equal) definition for a framed knot K is

$$Z(K) = e^{\Theta \cdot \mathrm{writhe}(K)/2} \cdot S\left(\tilde{Z}(K)\right) \in \mathcal{A} \subset \mathcal{A}',$$

  where Θ is the chord diagram , S is the standard algebra map $\mathcal{A}^r = \mathcal{A}/ <\Theta> \rightarrow \mathcal{A}$ defined by mapping Θ to 0 and leaving all other primitives of $\mathcal{A}$ in place, and $\tilde{Z}$ is the Kontsevich integral as in [Ko1].
- χ is the symmetrization map $\mathcal{B}'_\times \rightarrow \mathcal{A}'$, as on page 220. It is an algebra isomorphism by [B-N1] and the definition of $\times$.
- $\hat{\Omega}$ is the wheeling map as on page 223. We argue that it should be an algebra (iso-)morphism (Conjecture 2).
- $RT_\mathfrak{g}$ denotes the Reshetikhin–Turaev knot invariant associated with the Lie algebra $\mathfrak{g}$ [Re1, Re2, RT, Tu].
- $\mathcal{T}_\mathfrak{g}^\hbar$ (in all three instances) is the usual "diagrams to Lie algebras" map, as in [B-N1, Section 2.4 and exercise 5.1]. The only variation we make is that we multiply the image of a degree m element of $\mathcal{A}'$ (or $\mathcal{B}'_\times$ or $\mathcal{B}'_\cup$) by $\hbar^m$. In the construction of $\mathcal{T}_\mathfrak{g}^\hbar$ an invariant bilinear form on $\mathfrak{g}$ is needed. We use the standard form $(\cdot,\cdot)$ used in [RT] and in [CP, Appendix]. See also [Kac, Chapter 2].
- The isomorphism $\beta_\mathfrak{g}$ was already discussed when $S(\mathfrak{g})^\mathfrak{g}_\times[[\hbar]]$ was defined on page 226.
- The definition of the "Duflo map" $D(j_\mathfrak{g}^{1/2})$ requires some preliminaries. If V is a vector space, there is an algebra map D: $P(V) \rightarrow \mathrm{Diff}(V^\star)$ between the algebra $P(V)$ of polynomial functions on V and the algebra $\mathrm{Diff}(V^\star)$ of constant coefficients differential operators on the symmetric algebra $S(V)$. The map D is defined on generators as follows: If $\alpha \in V^\star$ is a degree 1 polynomial on V, set $D(\alpha)(v) = \alpha(v)$ for $v \in V \subset S(V)$, and extend $D(\alpha)$ to be a derivation on $S(V)$, using the Leibnitz law. A different (but less precise) way of defining D is via the Fourier transform: Think of $S(V)$ as a space of functions on $V^\star$. A polynomial function on V becomes a differential operator on $V^\star$ after taking the Fourier transform, and this defines our map D. Either way, if $j \in P(V)$ is homogeneous of degree k, the differential operator $D(j)$ lowers degrees by k and thus vanishes on the low degrees of $S(V)$. Hence $D(j)$ makes sense even when j is a power series instead of a polynomial. This definition has a natural extension to the case when the spaces involved are extended by $\mathbf{C}[[\hbar]]$, or even $\mathbf{C}((\hbar))$, the algebra

of Laurent polynomials in $\hbar$.

Now use this definition of D with $V = \mathfrak{g}$ to define the Duflo map $D(j_{\mathfrak{g}}^{1/2})$, where $j_{\mathfrak{g}}(X)$ is defined for $X \in \mathfrak{g}$ by

$$j_{\mathfrak{g}}(X) = \det\left(\frac{\sinh \operatorname{ad} X/2}{\operatorname{ad} X/2}\right).$$

The square root $j_{\mathfrak{g}}^{1/2}$ of $j_{\mathfrak{g}}$ is defined as in [Du] or [BGV, Section 8.2], and is a power series in X that begins with 1. We note that by Kirillov's formula for the character of the trivial representation (see e.g. [BGV, Theorem 8.4 with $\lambda = i\rho$]), $j_{\mathfrak{g}}^{1/2}$ is the Fourier transform of the symplectic measure on $M_{i\rho}$, where $M_{i\rho}$ is the co-adjoint orbit of $i\rho$ in $\mathfrak{g}_{\mathbf{R}}^{\star}$ (see e.g. [BGV, Section 7.5]):

$$(5) \qquad j_{\mathfrak{g}}^{1/2}(X) = \int_{r \in M_{i\rho}} e^{ir(X)} dr.$$

(We consider the symplectic measure as a measure on $\mathfrak{g}_{\mathbf{R}}^{\star}$, whose support is the subset $M_{i\rho}$ of $\mathfrak{g}_{\mathbf{R}}^{\star}$. Its Fourier transform is a function on $\mathfrak{g}_{\mathbf{R}}$ that can be computed via integration on the support $M_{i\rho} \subset \mathfrak{g}_{\mathbf{R}}^{\star}$ of the symplectic measure.) Duflo [Du, théorème V.2] (see also [Gi]) proved that $D(j_{\mathfrak{g}}^{1/2})$ is an algebra isomorphism.

- $\psi_{\mathfrak{g}}$ is the Harish-Chandra isomorphism $U(\mathfrak{g})^{\mathfrak{g}} \to P(\mathfrak{h}^{\star})^W$ extended by $\hbar$. Using the representation theory of $\mathfrak{g}$, it is defined as follows. If z is in $U(\mathfrak{g})^{\mathfrak{g}}$ and $\lambda \in \mathfrak{h}^{\star}$ is a positive integral weight, we set $\psi_{\mathfrak{g}}(z)(\lambda)$ to be the scalar by which z acts on the irreducible representation of $\mathfrak{g}$ whose highest weight is $\lambda - \rho$. It is well known (see e.g. [Hu, Section 23.3]) that this partial definition of $\psi_{\mathfrak{g}}(z)$ extends uniquely to a Weyl-invariant polynomial (also denoted $\psi_{\mathfrak{g}}(z)$) on $\mathfrak{h}^{\star}$, and that the resulting map $\psi_{\mathfrak{g}}\colon U(\mathfrak{g})^{\mathfrak{g}} \to P(\mathfrak{h}^{\star})^W$ is an isomorphism.
- The two equalities at the lower right quarter of the monster diagram need no explanation. We note though that if the space of polynomials $P(\mathfrak{g}^{\star})^{\mathfrak{g}}[[\hbar]]$ is endowed with its obvious algebra structure, only the lower equality is in fact an equality of algebras.
- $\iota_{\mathfrak{g}}$ is the restriction map induced by the identification of $\mathfrak{h}^{\star}$ with a subspace of $\mathfrak{g}^{\star}$ defined using the form $(\cdot,\cdot)$ of $\mathfrak{g}$. The map $\iota_{\mathfrak{g}}$ is an isomorphism by Chevalley's theorem (see e.g. [Hu, Section 23.1] and [BtD, Section VI-2]).
- $S_{\mathfrak{g}}$ is the extension by $\hbar$ of an integral operator. If $p(\lambda)$ is an invariant polynomial of $\lambda \in \mathfrak{g}^{\star}$, then

$$S_{\mathfrak{g}}(p)(\lambda) = \int_{r \in M_{i\rho}} p(\lambda - ir) dr.$$

$S_{\mathfrak{g}}$ can also be viewed as a convolution operator (with a measure concentrated on M_ρ), and like all convolution operators, it maps polynomials to polynomials.

2.3. THE FACES.

- The commutativity of the face labeled 1 was proven by Kassel [Kas] and Le and Murakami [LM1] following Drinfel'd [Dr1, Dr2]. We comment that it is this commutativity that makes the notion of "canonical Vassiliev invariants" [BG] interesting.
- The commutativity of the face labeled 2 is immediate from the definitions, and was already noted in [B-N1].
- The commutativity of the face labeled 3 (notice that this face fully encloses the one labeled 5) is due to Duflo [Du, théorème V.1].

PROPOSITION 2.1: *The face labeled* 4 *is commutative.*

Remark 2.2: Recalling that $D(j_{\mathfrak{g}}^{1/2})$ is an algebra isomorphism, this proposition becomes the precise formulation of Theorem 2.

Proof of Proposition 2.1: Follows immediately from the following two lemmas, taking $C = \Omega$ in (6). ∎

LEMMA 2.3: *Let κ: $\mathfrak{g} \to \mathfrak{g}^\star$ be the identification induced by the standard bilinear form $(\cdot,\cdot)$ of $\mathfrak{g}$. Extend κ to all symmetric powers of $\mathfrak{g}$, and let $\kappa^\hbar$: $S(\mathfrak{g})^{\mathfrak{g}}[[\hbar]] \to S(\mathfrak{g}^\star)((\hbar))$ be defined for a homogeneous $s \in S(\mathfrak{g})^{\mathfrak{g}}[[\hbar]]$ (relative to the grading of $S(\mathfrak{g})$) by $\kappa^\hbar(s) = \hbar^{-\deg s}\kappa(s)$. If $C \in \mathcal{B}'$ is a uni-trivalent diagram, $\hat{C} : \mathcal{B}' \to \mathcal{B}'$ is the operator corresponding to C as in Definition 1.1, and $C' \in \mathcal{B}'$ is another uni-trivalent diagram, then*

$$\mathcal{T}_{\mathfrak{g}}^\hbar \hat{C}(C') = D(\kappa^\hbar \mathcal{T}_{\mathfrak{g}}^\hbar C)\mathcal{T}_{\mathfrak{g}}^\hbar C'. \tag{6}$$

Proof: If κj is a tensor in $S^k(\mathfrak{g}^\star) \subset \mathfrak{g}^{\star\otimes k}$, the k'th symmetric tensor power of $\mathfrak{g}^\star$, and j' is a tensor in $S^{k'}(\mathfrak{g}) \subset \mathfrak{g}^{\otimes k'}$, then

$$D(\kappa j)(j') = \begin{cases} 0 & \text{if } k > k', \\ \begin{array}{l}\text{the sum of all ways of contracting all}\\ \text{the tensor components of } j \text{ with some}\\ \text{(or all) tensor components of } j'\end{array} & \text{otherwise.}\end{cases} \tag{7}$$

By definition, the "diagrams to Lie algebras" map carries gluing to contraction, and hence carries the operation in Definition 1.1 to the operation in (7), namely, to D. Counting powers of $\hbar$, this proves (6). ∎

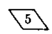
LEMMA 2.4: $\kappa^\hbar \mathcal{T}_\mathfrak{g}^\hbar \Omega = j_\mathfrak{g}^{1/2}$.

Proof: It follows easily from the definition of $\mathcal{T}_\mathfrak{g}^\hbar$ and of κ^h that $(\kappa^\hbar \mathcal{T}_\mathfrak{g}^\hbar \omega_n)(X) = \operatorname{tr}(\operatorname{ad} X)^n$ for any $X \in \mathfrak{g}$. Hence, using the fact that $\kappa^\hbar \circ \mathcal{T}_\mathfrak{g}^\hbar$ is an algebra morphism if $\mathcal{B}'$ is taken with the disjoint union product,

$$\begin{aligned}(\kappa^\hbar \mathcal{T}_\mathfrak{g}^\hbar \Omega)(X) &= \exp \sum_{n=1}^\infty b_{2n} (\kappa^\hbar \mathcal{T}_\mathfrak{g}^\hbar \omega_{2n})(X) \\ &= \exp \sum_{n=1}^\infty b_{2n} \operatorname{tr}(\operatorname{ad} X)^{2n} = \det \exp \sum_{n=1}^\infty b_{2n} (\operatorname{ad} X)^{2n}.\end{aligned}$$

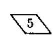
By the definition of the modified Bernoulli numbers (2), this is

$$\det \exp \frac{1}{2} \log \frac{\sinh \operatorname{ad} X/2}{\operatorname{ad} X/2} = \det \left(\frac{\sinh \operatorname{ad} X/2}{\operatorname{ad} X/2} \right)^{1/2} = j_\mathfrak{g}^{1/2}(X). \qquad \blacksquare$$

PROPOSITION 2.5: *The face labeled*  *is commutative.*

Proof: According to M. Vergne (private communication), this is a well known fact, but we could only find it in [Gi], where the language is somewhat different. For completeness, we present a short proof here, albeit with some analytical details omitted. We first modify the statement in three minor ways:

- We ignore the extension of all spaces involved by $\hbar$. This extension, needed in some other parts of this paper, makes no difference when it comes to  .
- We strengthen the statement slightly by dropping the $\mathfrak{g}$ invariance restriction from all spaces involved.
- Instead of working with the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$ and the (equivalent) algebra $P(\mathfrak{g}^\star)$ of polynomials on $\mathfrak{g}^\star$, we switch to working with the space $F_\epsilon(\mathfrak{g}_\mathbf{R}^\star)$ of polynomials in $\alpha \in \mathfrak{g}_\mathbf{R}^\star$ multiplied by the Gaussian $e^{-\epsilon|\alpha|^2}$.

It is clear that both $S_\mathfrak{g}$ and $D(j_\mathfrak{g}^{1/2})$ are defined on $F_\epsilon(\mathfrak{g}_\mathbf{R}^\star)$, and that the equality $D(j_\mathfrak{g}^{1/2}) = S_\mathfrak{g}$ on $F_\epsilon(\mathfrak{g}_\mathbf{R}^\star)$ would imply the commutativity of  after taking the $\epsilon \to 0$ limit. On the other hand, the functions in $F_\epsilon(\mathfrak{g}_\mathbf{R}^\star)$ are smooth and rapidly decreasing, and hence the tools of Fourier analysis are available.

We now prove the equality $D(j_\mathfrak{g}^{1/2}) = S_\mathfrak{g}$ on $F_\epsilon(\mathfrak{g}_\mathbf{R}^\star)$. Conjugating by the Fourier transform (over $\mathfrak{g}_\mathbf{R}^\star$), the differential operator $D(j_\mathfrak{g}^{1/2})$ becomes the operator of multiplication by $j_\mathfrak{g}^{1/2}(iX)$ on the space of rapidly decreasing smooth functions on $\mathfrak{g}_\mathbf{R}$ (recall that in general the Fourier transform takes $\partial/\partial x$ to multiplication by ix). Conjugating by the inverse Fourier transform, we see that

$D(j_{\mathfrak{g}}^{1/2})$ is the operator of convolution with the inverse Fourier transform of $j_{\mathfrak{g}}^{1/2}(iX)$ (recall that the Fourier transform intertwines between multiplication and convolution), which is the symplectic measure on M_ρ (see (5)). So $D(j_{\mathfrak{g}}^{1/2})$ is convolution with that measure, as required. ∎

3. Proof of Theorem 1

We prove the slightly stronger equality

$$\mathcal{T}_{\mathfrak{g}}^{\hbar}\Omega = \mathcal{T}_{\mathfrak{g}}^{\hbar}\chi^{-1}Z(\bigcirc). \tag{8}$$

Proof: We compute the right hand side of (8) by first computing $S_{\mathfrak{g}}\iota_{\mathfrak{g}}{}^{-1}\psi_{\mathfrak{g}}RT_{\mathfrak{g}}(\bigcirc)$ and then using the commutativity of the monster diagram. It is known (see e.g. [Cp, example 11.3.10]) that if $\lambda - \rho \in \mathfrak{h}^\star$ is the highest weight of some irreducible representation $R_{\lambda-\rho}$ of $\mathfrak{g}$, then

$$(\psi_{\mathfrak{g}}RT_{\mathfrak{g}}(\bigcirc))(\lambda) = \frac{1}{\dim R_{\lambda-\rho}} \prod_{\alpha\in\Delta_+} \frac{\sinh \hbar(\lambda,\alpha)/2}{\sinh \hbar(\rho,\alpha)/2},$$

where Δ_+ is the set of positive roots of $\mathfrak{g}$ and $(\cdot,\cdot)$ is the standard invariant bilinear form on $\mathfrak{g}$. By the Weyl dimension formula and some minor arithmetic, we get (see also [LM2, section 7])

$$(\psi_{\mathfrak{g}}RT_{\mathfrak{g}}(\bigcirc))(\lambda) = \prod_{\alpha\in\Delta_+} \frac{\hbar(\rho,\alpha)/2}{\sinh \hbar(\rho,\alpha)/2} \cdot \frac{\sinh \hbar(\lambda,\alpha)/2}{\hbar(\lambda,\alpha)/2}. \tag{9}$$

We can identify $\mathfrak{g}$ and $\mathfrak{g}^\star$ using the form $(\cdot,\cdot)$, and then expressions like '$\operatorname{ad}\lambda$' make sense. By definition, if $\mathfrak{g}_\alpha$ is the weight space of the root α, then $\operatorname{ad}\lambda$ acts as multiplication by (λ,α) on $\mathfrak{g}_\alpha$, while acting trivially on $\mathfrak{h}$. From this and (9) we get

$$\begin{aligned}(\psi_{\mathfrak{g}}RT_{\mathfrak{g}}(\bigcirc))(\lambda) &= \det\left(\frac{\operatorname{ad}\hbar\rho/2}{\sinh\operatorname{ad}\hbar\rho/2}\right)^{1/2} \cdot \det\left(\frac{\sinh\operatorname{ad}\hbar\lambda/2}{\operatorname{ad}\hbar\lambda/2}\right)^{1/2} \\ &= j_{\mathfrak{g}}^{-1/2}(\hbar\rho)\cdot j_{\mathfrak{g}}^{1/2}(\hbar\lambda).\end{aligned}$$

The above expression (call it $Z(\lambda)$) makes sense for all $\lambda \in \mathfrak{g}^\star$, and hence it is also $\iota_{\mathfrak{g}}{}^{-1}\psi_{\mathfrak{g}}RT_{\mathfrak{g}}(\bigcirc)$. So we're only left with computing $S_{\mathfrak{g}}Z(\lambda)$:

$$S_{\mathfrak{g}}Z(\lambda) = \int_{r\in M_{i\rho}} dr\, Z(\lambda - ir) = j_{\mathfrak{g}}^{-1/2}(\hbar\rho)\int_{r\in M_{i\rho}} dr\, j_{\mathfrak{g}}^{1/2}(\hbar(\lambda - ir)).$$

By (5), this is

$$j_{\mathfrak{g}}^{-1/2}(\hbar\rho)\int_{r\in M_{i\rho}} dr \int_{r'\in M_{i\rho}} dr'\, e^{i\hbar(r',\lambda-ir)}$$
$$= j_{\mathfrak{g}}^{-1/2}(\hbar\rho)\int_{r'\in M_{i\rho}} dr'\, e^{i\hbar(r',\lambda)}\int_{r\in M_{i\rho}} dr\, e^{i\hbar(-ir',r)}.$$

Using (5) again, we find that the inner-most integral is equal to $j_{\mathfrak{g}}^{1/2}(\hbar\rho)$ independently of r', and hence

$$S_{\mathfrak{g}}Z(\lambda) = \int_{r'\in M_{i\rho}} dr'\, e^{i\hbar(r',\lambda)},$$

and using (5) one last time we find that

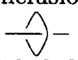
$$S_{\mathfrak{g}}Z(\lambda) = j_{\mathfrak{g}}^{1/2}(\hbar\lambda). \tag{10}$$

The left hand side of (8) was already computed (up to duality and powers of $\hbar$) in Lemma 2.4. Undoing the effect of $\kappa^{\hbar}$ there, we get the same answer as in (10). ∎

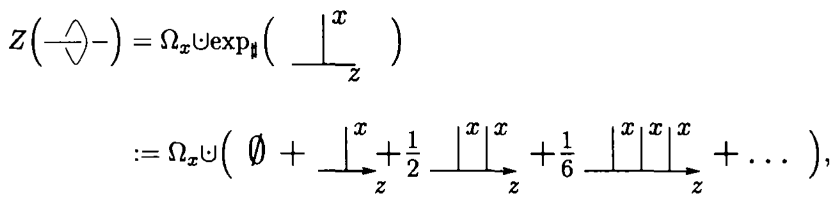
4. Epilogue

After the first version of this paper was circulating, Kontsevich [Ko2] proved the Wheeling Conjecture (Conjecture 2) using the 2-dimensional configuration-space techniques he developed for the proof of his celebrated "Formality Conjecture". At that time it was already known to DPT and T. Q. T. Le (see [BLT]) that the Wheeling Conjecture implies the Wheels Conjecture (Conjecture 1), and thus both conjectures were known to be true, though the proof of the implication Wheeling $\Rightarrow$ Wheels did not shed light on the fundamental relationship that ought to exist between the two conjectures (see Remark 1.4).

In the summer of 1998, DBN and DPT found a knot-theoretic proof of the Wheeling Conjecture, which also sheds some more light on the relationship between it and the Wheels Conjecture. We sketch these results here; the details will appear in [BLT]. We only present an idealized picture, in which a single theorem, Theorem 4 below, implies both conjectures. We admit that the truth is somewhat less clean: the proof of Theorem 4 in [BLT] involves a bootstrap procedure that uses some results from this paper (at least implicitly) and in which the conclusions, Wheels and Wheeling, are proven first.

Let  denote the **long Hopf link**: the usual Hopf link, with one component, labeled z, "opened up", and with the other component, labeled x, presented by a round circle, so that the result looks precisely like its symbol. As well known, the framed Kontsevich integral has an extension to links, and when evaluated on , it is valued in a space of diagrams $\mathcal{A}'$ similar to $\mathcal{A}$, only that each diagram in $\mathcal{A}'$ has two skeleton components: a line labeled z and a circle labeled x. We then use a diagrammatic PBW theorem, similar to the one in (3), to map $\mathcal{A}'$ to a space $\mathcal{A}''$ in which the x part of the skeleton is replaced by an unordered set of x-marked univalent vertices.

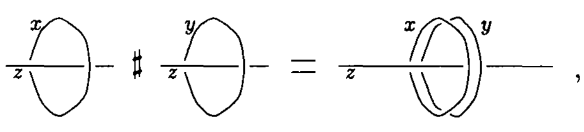
THEOREM 4 (See [BLT]): *In $\mathcal{A}''$,*



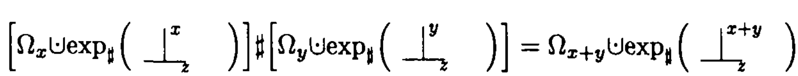
where Ω_x denotes Ω with all univalent vertices marked x, and $\emptyset$ denotes the empty (unit) diagram.

It is clear that Theorem 4 and the simple behavior of the Kontsevich integral with respect to dropping a link component implies the Wheels Conjecture. Simply drop the component labeled z from the left hand side of (11), and the skeleton component labeled z from the right hand side of that equation. What remains is precisely the Wheels Conjecture.

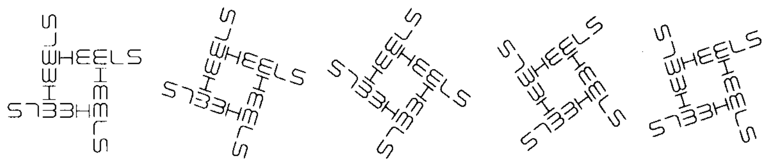
The proof of the Wheeling Conjecture from Theorem 4 is elegant but a bit more involved. We start from the following $1 + 1 = 2$ equality of links,

(11) 

which says that the connected sum of two copies of is equal to the same with the x-component doubled. The Kontsevich integral behaves nicely with respect to the operations of connected sum [B-N1, Kol] and of doubling [LM2], and hence by computing the Kontsevich integral of both sides of (11), one can translate that equation to an equality between two sums of diagrams, presented schematically as



As explained in [BLT], this equality is combinatorially equivalent to the Wheeling Conjecture.



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